Noise and Signal Transfer Functions for DfMUX Mixers

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Abstract

The DfMUX demodulator consists of several coherent AM receivers operating on the same input signal, but with different carrier frequencies. For each channel, a mixer is required. In this memo, several mixer designs (sinusoidal, half-wave square, and quarter-wave square mixers) are investigated. We derive the transfer functions for each mixer in response to both the modulated signal and additive white noise.

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1 Background

The DfMUX demodulator (DMFD) consists of a number of synchronous AM receivers, all tuned to different carrier frequencies on the same input signal. For our purposes, it suffices to consider a single-channel AM receiver, as shown in Figure 1.

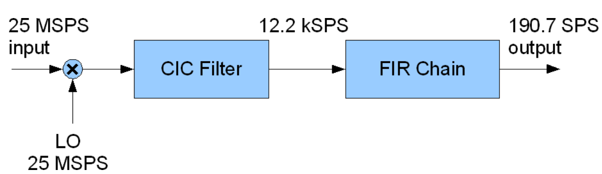

Figure 1: DfMUX Signal Flow

Each of these blocks are implemented using sampled signals. It is much simpler to investigate a continuous-time model; the resulting approximation is very good provided the oversampling rate is sufficiently high. The continuous-time model is shown in Figure 2.

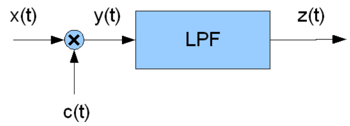

Figure 2: Ideal Signal Flow

The mixer, which consists of a carrier signal and multiplier block, is shown on the left of Figure 2. Its purpose is to down-convert (heterodyne) the AM input into a baseband signal. It is followed by a low-pass filter bank which removes all out-of-band components and leaves only the baseband.

We are interested in the noise and signal components present in the mixer output $y(t)$. The presence of the low-pass filter allows us to consider $y(t)$ in a relaxed manner, since out-of-band components will be removed and may be disregarded.

1.1 Signal Model

In this memo, we admit signals $x(t)$ of the following form:

$$x(t) = s(t)\cos(\omega t + \theta) + n(t) \tag{1}$$

where $s(t)$ is the signal we wish to recover and $n(t)$ is an additive noise term.

1.2 Noise Models

We consider two alternatives for the noise signal $n(t)$.

First, we consider a broadband white noise source $n_w(t)$ with variance σ_n^2. This noise source may be used to model instrumentation noise occurring in the signal flow after the bandpass filter that defines each receiver channel.

Next, we consider a narrowband noise signal $n_n(t)$. We assume $n_n(t)$ is centered about the radian frequency σ, has a bandwidth approximately equal to that of the signal $x(t)$, and has variance σ_n^2. It is not necessary to formally parameterize $n_n(t)$ beyond making a few simple assumptions about its bandwidth. This noise signal may be used to model noise in the received signal, or instrumentation noise prior to the bandpass filter defining each receiver channel. We expand $n_n(t)$ using Rice's representation:

$$n_n(t) = n_i(t)\cos(\sigma t) + n_q(t)\sin(\sigma t)$$

where the in-phase and quadrature processes ($n_i(t)$ and $n_q(t)$ respectively) are zero-mean, lowpass processes with variance σ_n^2.

1.3 Methodology

We consider the system response due to noise and signal components separately. That is, we investigate the signal-related output $y_s(t)$ when $n(t) = 0$, and noise-related output $y_n(t)$ when $s(t) = 0$. We do this in the hopes that the overall output $y(t)$ is approximately the sum of these individual contributions, i.e. $y(t) \approx y_n(t) + y_s(t)$.

Although we appear to invoke the superposition principle, strictly speaking, superposition does not apply to nonlinear systems. In particular, we ignore a great deal of mixing between signal and noise components. This is only permissible only as long as the noise amplitude remains low enough that these mixer products are negligible.

In the following sections, we consider each mixer type separately. We derive output-to-input ratios for the signal and both noise components. We denote the signal and noise transfer ratios as R_s and R_n respectively.

2 Sinusoidal Mixer

In this section, we consider the carrier signal of the following form:

$$c(t) = A_c \cos \sigma t$$

Note that the amplitude term A_c is a peak amplitude, not RMS. Where RMS units are required for the carrier signal, we define $A_c^{\mathrm{RMS}} = A_c\sqrt{2}/2$. The mixer output $y(t)$ is given as follows:

$$\begin{aligned} y(t) &= x(t)c(t) \\ &= A_c \left[s(t)\cos(\omega t + \theta) + n(t)\right]\cos(\sigma t) \\ &= \frac{A_c}{2}s(t)\left[\cos[(\omega - \sigma)t]\cos\theta - \sin[(\omega - \sigma)t]\sin\theta\right] + A_c n(t)\cos(\sigma t) + \quad \ldots \end{aligned}$$

where the ellipsis (...) signifies components centered at the frequency $\pm(\omega + \sigma)$ which will be attenuated by the LPF and need not be considered.

2.1 Signal Component

With noise neglected, the signal-dependent, baseband portion of $y(t)$ is given by the following:

$$y_s(t) = \frac{A_c}{2}s(t)\left[\cos\left[(\omega - \sigma)\, t\right]\cos\theta - \sin\left[(\omega - \sigma)\, t\right]\sin\theta\right]$$

We now list a few special cases depending on ω, σ, and θ.

Frequency-Locked, Phase-Locked

When $\theta = 0$ and $\omega = \sigma$, the signal-dependent output component is given by:

$$y_s(t) = \frac{A_c}{2}s(t)$$

The signal transfer function is:

$$R_s = A_c/2 = A_c^{\mathrm{RMS}}\sqrt{2}/2.$$

Frequency-Locked, Constant Phase Offset

When $\theta \neq 0$ and $\omega = \sigma$, the signal-dependent output component is given by:

$$\frac{A_c}{2}s(t)\cos\theta$$

The signal is attenuated by a phase-dependent factor, and the signal transfer function is

$$R_s = \frac{A_c\cos\theta}{2} = \frac{\sqrt{2}}{2}A_c^{\mathrm{RMS}}\cos\theta.$$

Constant Frequency Offset

When $\omega \neq \sigma$, the phase parameter θ represents an initial phase offset which may arbitrarily chosen. For convenience, we select $\theta = 0$, and the output is given by:

$$y_s(t) = \frac{A_c}{2} s(t) \cos\left[(\omega - \sigma)\, t\right]$$

The transfer ratio is time-dependent due to beating at the radian frequency $\omega - \sigma$:

$$R_s(t) = \frac{A_c}{2} \cos\left[(\omega - \sigma)\, t\right] = \frac{A_c^{\text{RMS}}\sqrt{2}}{2} \cos\left[(\omega - \sigma)\, t\right]$$

2.2 Broadband Noise Component

Neglecting the signal component $s(t)$, the noise-dependent response becomes:

$$y_n(t) = A_c n(t) \cos \sigma t$$

We wish to derive the variance σ_y^2 of $y_n(t)$. Because both $y_n(t)$ is zero-mean, it suffices to consider the second-order moment.

$$\sigma_y^2 = E\{y_n(t)^2\} = A_c^2 E\{n(t)^2 \cos^2 \sigma t\} = A_c^2 \sigma_n^2 / 2$$

The transfer function for noise, in RMS volts out per RMS volt in, is:

$$R_n = \frac{\sigma_y}{\sigma_n} = \frac{A_c\sqrt{2}}{2} = A_c^{\text{RMS}}$$

2.3 Narrowband Noise Component

When $n(t) = n_n(t) = n_i(t)\cos(\sigma t) + n_q(t)\sin(\sigma t)$, we may view the noise signal as a quadrature amplitude-modulated (QAM) signal. Assuming frequency and phase lock and neglecting out-of-band terms, the output $y(t)$ is given as follows:

$$y(t) = \frac{A_c}{2} n_i(t) + \quad \ldots$$

To determine the RMS noise voltage, we note that $y(t)$ is zero-mean and take the second moment:

$$E\{y^2(t)\} = \left[\frac{A_c}{2}\right]^2 E\{n_i^2(t)\} = \left[\frac{A_c}{2}\right]^2 \sigma_n^2$$

The output (in volts RMS) is:

$$\frac{A_c \sigma_n}{2} = \frac{A_c^{\text{RMS}}\sqrt{2}\sigma_n}{2}$$

The transfer function is given by:

$$R_n = \frac{A_c}{2} = \frac{A_c^{\text{RMS}}\sqrt{2}}{2}$$

3 Half-Wave (Square) Mixer

The half-wave carrier signal $c_h(t)$ is shown in Figure 3.

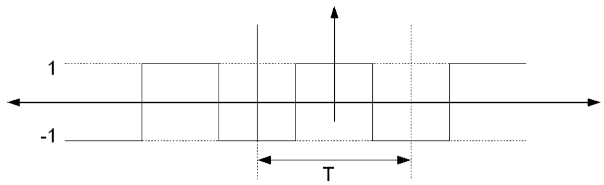

Figure 3: Half-wave Carrier Signal

3.1 Signal Component

When considering the signal component, it is convenient to use the Fourier series expansion of the carrier signal. We have:

$$c_h(t) = \sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t}$$

We are only interested in the baseband components, since all other mixer-signal products are removed by the low-pass filter — that is, we need only determine c_{-1}, c_0, and c_1. Clearly, $c_h(t)$ has no DC component; therefore, $c_0 = 0$. It can be shown that $c_{-1} = c_1 = 2/\pi$. Thus, the mixer may be expanded as follows:

$$c_h(t) = \frac{2}{\pi} e^{-i2\pi t/T} + \frac{2}{\pi} e^{i2\pi t/T} + \ \ldots$$

equivalently,

$$c_h(t) = \frac{4}{\pi} \cos\left(\frac{2\pi t}{T}\right) + \ \ldots$$

This is the same as the sinusoidal carrier when $A_c = 4/\pi$. When phase- and frequency-locked, the output is given by:

$$y(t) = \frac{2}{\pi} s(t)$$

The input-to-output transfer ratio is:

$$R_s = 4/\pi \times 1/2 = 2/\pi.$$

3.2 Broadband Noise Component

The noise component does not require the Fourier series expansion, since the half-wave mixer simply multiplies uncorrelated white noise by either $+1$ or -1. The output noise remains uncorrelated with the same variance; thus, the noise transfer function is 1.

3.3 Narrowband Noise Component

We adopt the same approach used when considering the signal component, using Rice's representation for the noise. The output $y(t)$ is given as:

$$y(t) = \frac{2}{\pi} n_q(t)$$

The RMS output is:

$$\frac{2}{\pi} \sigma_n$$

Thus,

$$R_n = \frac{2}{\pi}$$

4 Quarter-Wave Mixer

The quarter-wave carrier signal $c_q(t)$ is shown in Figure 4.

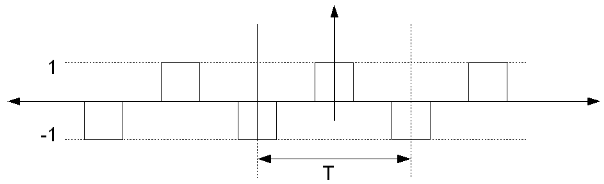

Figure 4: Quarter-wave Carrier Signal

4.1 Signal Component

Once again, it is convenient to derive a Fourier-series representation for the quarter-wave noise. As above, we have the representation:

$$c_q(t) = \sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t}$$

where only c_{-1} and c_1 are relevant. In this case, it can be shown that $c_{-1} = c_1 = \frac{\sqrt{2}}{\pi}$; thus,

$$c_q(t) = \frac{2\sqrt{2}}{\pi} \cos\left(\frac{2\pi t}{T}\right) + \ \dots$$

Once again, we may use the sinusoidal model with an amplitude of $A_c = 2\sqrt{2}/\pi$; the signal-dependent transfer function is $\sqrt{2}/\pi$.

4.2 Broadband Noise Component

The quarter-wave mixer multiplies the noise signal $n(t)$ by 1, 0, or -1, depending on its phase. As with the half-wave mixer, a gain of -1 is essentially identical to a gain of 1 when the noise is white. Thus, we may invoke Parseval's theorem and note that the output noise power is 1/2 of the input noise power. We have $\sigma_y^2 = \sigma_n^2/2$, and thus the noise transfer function (in volts RMS) is:

$$R_n = \frac{\sigma_y}{\sigma_n} = \frac{\sqrt{2}}{2}$$

4.3 Narrowband Noise Component

As before, we adopt Rice's representation for the noise. The output $y(t)$ is given as:

$$y(t) = \frac{\sqrt{2}}{\pi} n_q(t)$$

The RMS output is:

$$\frac{\sqrt{2}}{\pi}\sigma_n$$

The transfer function is:

$$R_n = \frac{\sqrt{2}}{\pi}$$

5 Spur Response

In testing the DfMUX demodulator, it is convenient to examine the system's response to a single sinusoidal tone with a small offset from the carrier signal. In this section, we examine the demodulator output when the input signal is of the form:

$$x(t) = A_s \cos(\omega t)$$

We have, using the sinusoidal mixer as our model:

$$\begin{aligned} y(t) &= A_s A_c \cos(\omega t) \cos(\sigma t) \\ &= \frac{A_s A_c}{2} \left[\cos(\omega - \sigma)t + \cos(\omega + \sigma)t \right] \end{aligned}$$

If the frequency $\omega + \sigma$ is out-of-band, we may neglect the associated components. In this case, we have:

$$z(t) = \frac{A_s A_c}{2} \left[\cos(\omega - \sigma)t \right]$$

The half- and quarter-wave square mixers may be considered by replacing A_c with their first Fourier series amplitudes. We summarize the amplitude in Table 1.

Units	Sinusoidal Mixer	Half-wave Mixer	Quarter-wave Mixer
Absolute	$\frac{A_s A_c}{2} \cos(\omega - \sigma)t$	$\frac{2}{\pi} A_s \cos(\omega - \sigma)t$	$\frac{\sqrt{2}}{\pi} A_s \cos(\omega - \sigma)t$
RMS	$A_s^{\mathrm{RMS}} A_c^{\mathrm{RMS}} \cos(\omega - \sigma)t$	$\frac{2\sqrt{2}}{\pi} A_s^{\mathrm{RMS}} \cos(\omega - \sigma)t$	$\frac{2}{\pi} A_s^{\mathrm{RMS}} \cos(\omega - \sigma)t$

Table 1: Summary of spur response amplitudes.

6 Summary

Table 2 summarizes the transfer-function results derived above. These results assume correct phase- and frequency-locking.

Mixer	Signal	Broadband Noise	Narrowband Noise
Sinusoidal	$A_c/2 = \frac{\sqrt{2}}{2}A_c^{\mathrm{RMS}}$	$A_c\sqrt{2}/2 = A_c^{\mathrm{RMS}}$	$\frac{A_c}{2} = \frac{A_c^{\mathrm{RMS}}\sqrt{2}}{2}$
Half-Wave (Square)	$2/\pi$	1	$2/\pi$
Quarter-Wave	$\sqrt{2}/\pi$	$\sqrt{2}/2$	$\sqrt{2}/\pi$

Table 2: Transfer functions and signal-to-noise transfer ratios for DfMUX mixer models.